Mathematics 222B Lecture 15 Notes

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1 Unique Continuation for Elliptic PDEs and Introduction to Hyperbolic PDEs

1.1 Unique continuation for elliptic PDEs

The original plan was for this lecture to cover one final topic for elliptic PDEs: unique continuation. Here is the main theorem.

Theorem 1.1 (Aronszajn). *Let $U \subseteq \mathbb{R}^d$ be open and connected, and consider the elliptic partial differential operator P with*

$$Pu = -\partial_j(a^{j,k}\partial_k u) + b^j \partial_j u + cu,$$

where $a^{j,k}b^j, c \in C^\infty(U)$ with $a \succ \lambda I$ in U. Let $u \in H^1(U)$. If $Pu = 0$ in U and $u = 0$ in a nonempty open subset $Q \subseteq U$, then $u = 0$ in U.

For holomorphic functions, the way we prove this is to say that holomorphic functions are analytic and look at the domain of convergence of a Taylor series. The way we prove this for solutions to elliptic PDEs is via an a priori estimate.

Lemma 1.1 (Carleman estimate). *Let $v \in C_c^\infty(\mathbb{R}^d)$. and suppose that $\nabla \psi \neq 0$. Then*

$$t^2 \|e^{t\psi} v\|_{L^2} + t\|e^{t\psi}\nabla v\|_{L^2} \leq C\|e^{t\psi} Pv\|_{L^2}.$$

A good reference for this is the book *Carlesman Estimates* by Lerner. This is related to inverse problems and other non-well-posed problems in PDEs.

1.2 Linear hyperbolic PDEs

Instead of formally defining what a hyperbolic PDE is, which is difficult and not entirely productive. Instead, we will give a 'working definition" of how people think of hyperbolic PDEs.

Definition 1.1. A **hyperbolic PDE** is an evolutionary PDE with two characteristics:

- "#/order of time derivatives" = "#/order of space derivatives".
- (local) well-posedness of the initial value problem

$$\begin{cases} P\phi = 0 \\ (\phi, \partial_t \phi, \dots, \partial_t^{N-1}\phi)|_{t=0} = (g_0, \dots, g_{N-1}), \end{cases}$$

  where N is the order of the time derivatives.

This second condition is really what people think of when they talk about hyperbolic PDEs.

Example 1.1. The wave equation $(-\partial_t^2 + \Delta)\phi = 0$ is hyperbolic.

Example 1.2. The equation $(-\partial_t + x^j \partial_j)\phi = 0$ is hyperbolic.

Example 1.3 (Non-examples)**.** The heat equation $(\partial_t - \Delta)\phi = 0$ and the Schrödinger equation $(\partial_t - i\Delta)\phi = 0$ are *dispersive* but not hyperbolic.

Example 1.4. The Laplace equation $(\partial_t^2 + \Delta)\phi = 0$ is not hyperbolic because it does not have local well-posedness of the initial value problem.

Local well-posedness of the initial value problem is related to the energy estimate.

Example 1.5 (Linear constant coefficient system)**.** Let

$$\Phi = \begin{bmatrix} \Phi^{(1)} \\ \vdots \\ \Phi^{(n)} \end{bmatrix},$$

and suppose we have a system of linear, constant coefficient PDEs

$$B\partial_t \Phi = A^j \partial_{x^j} \Phi,$$

where A is an $n \times n$ matrix. Without loss of generality, assume we have

$$\partial_t \Phi = A^j \partial_{x^j} \Phi,$$

What guarantees uniqueness of a solution to the initial value problem? That is, what condition do we need on A to guarantee the validity of the energy estimate?

$$\int_{\mathbb{R}^d} \underbrace{\Phi^{(k)} \partial_t \Phi^{(k)}}_{=\frac{1}{2}\partial_t \int \Phi^{(k)}\Phi^{(k)}} + \underbrace{\Phi^{(k)} (A^j)^{(k)}_{(\ell)} \partial_j \Phi^{(\ell)}}_{\frac{1}{2}\int (A^j)^{(k)}_{(\ell)} \Phi^{(k)} \partial_j \Phi^{(\ell)} - \frac{1}{2}(A^j)^{(k)}_{(\ell)} \partial_j \Phi^{(k)} \Phi^{(\ell)}} \, dx = \int \Phi^{(k)} F^{(k)} \, dx.$$

We get the following identity:

$$\frac{1}{2}\partial_t \int |\Phi|^2 \, dx + \frac{1}{2}\int ((A^j)^{(k)}_{(\ell)} - (A^j)^{(\ell)}_{(k)})\Phi^{(k)}\partial_j \Phi^{(\ell)} \, dx = \int F \cdot \Phi \, dx,$$

where the second term is 0 if A^j is symmetric.

This tells us that if A^j is symmetric, then the energy estimate holds:

$$\int |\Phi|^2(t) \, dx = \int |\Phi|^2(0) \, dx + \int_0^t \int F \cdot \Phi \, dx \, dt.$$

This gives uniqueness

Theorem 1.2. *The linear, constant coefficient system*

$$\partial_t \Phi = A^j \partial_{x^j} \Phi$$

is hyperbolic if and only if the A^j are symmetric. That is the initial value problem is well-posed in L^2, meaning for every $\Phi_0 \in L^2(\mathbb{R}^d)$,and $F \in L^1_t((-\infty, \infty); L^2_x)$, there exists a unique $\Phi \in C_t((-\infty, \infty); L^2_x)$ solving the system.

We use the notation $\phi \in C_t(I; X)$ to mean that the function $\phi : I \to X$ sending $t \mapsto \phi(t)$ is continuous, where $C_t(I; X)$ has the norm

$$\|\phi\|_{C_t(I;X)} := \sup_{t \in I} \|\phi(t, \cdot)\|_X = \|\phi\|_{L^\infty_t(X)} < \infty.$$

Example 1.6 (1st order formulation of $\Box\phi = f$)**.** Let the **d'Alembertian** be $\Box = -\partial_t^2 + \Delta$. Then

$$\Box\phi = f \iff \partial_t \phi = \psi, \partial_t \psi = \Delta\phi - f.$$

We can write this system as

$$\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

If we take the Fourier transform of the matrix, we get

$$\begin{bmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{bmatrix}.$$

and if we diagonalize this, we get

$$\begin{bmatrix} +i|\xi| & 0 \\ 0 & -i|\xi| \end{bmatrix},$$

which is anti-Hermitian. This means that the energy estimate will hold in the diagonalized variables

1.3 Goals for studying hyperbolic PDEs

Here are our goals for studying hyperbolic PDEs:

1. (Local) well-posedness of the initial value problem for variable-coefficient wave equations,
$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi,$$
where g is a **Lorentzian metric**, a non-degenerate symmetric $(d+1)\times(d+1)$ matrix with signature $(-,+,+\cdots,+)$ (meaning that the eigenvalues of g have signs $-,+,+\dots,+$). This condition can also be stated as: for every (t,x), there exists an invertible matrix M such that $M^{-1}g(t,x)M = \operatorname{diag}(-1,+1,+1,\dots,+1)$.

   Example 1.7. When $g = \operatorname{diag}(-1,+1,+1,\dots,+1)$ and $b = c = 0$, $P = \Box$.

2. Long-time behavior of the solutions: If we look at this in general, it immediately becomes a research topic.[1] Instead, we will focus on long-time behavior of solutions to equations where P is a small variant of $\Box$.

1.4 Grönwall's inequality

Our treatment for the well-posedness of the initial value problem for variable coefficient wave equations will be closer to Ringström's book *the Cauchy Problem in General Relativity* than it will be to Evans' book.

Our setting is
$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi.$$
We want to derive energy estimates for
$$\begin{cases} P\phi = f & \text{in } \mathbb{R}_+\times\mathbb{R}^d, \\ (\phi,\partial_t\phi)|_{t=0} = (g,h) & \text{on } \{t=0\}\times\mathbb{R}^d. \end{cases}$$

We need the following preliminary tool, which was discussed in Math 222A.

Lemma 1.2 (Grönwall's inequality)**.** *Suppose that $E(t)\in C_t([0,T])$ and $r(t)\in L^1_t([0,T])$ with $E,r\geq 0$ satisfy the inequality*
$$E(t) \leq E_0 + \int_0^t r(t')E(t')\,dt' \qquad \forall 0\leq t\leq T.$$
Then
$$E(t) \leq E_0\exp\left(\int_0^t r(t')\,dt'\right) \qquad \forall 0\leq t\leq T.$$

[1] Scattering theory is devoted to studying these problems.

We give a proof that uses a **bootstrap argument**, i.e. continuous induction on time. First, here is a motivating computation: Take the inequality we are given, and plug in the answer into the right hand side. We get

$$E(t) \le E_0 + E_0 \int_0^t r(t') \exp\Bigg(\underbrace{\int_0^{t'} r(t'')\, dt''}_{R(t')} \Bigg)\, dt,$$

where R is just an antiderivative of r.

$$= E_0 + E_0 \left(\exp\left(\int_0^t r(t')\, dt' \right) - 1 \right)$$

$$= E_0 \exp\left(\int_0^t r(t')\, dt' \right).$$

This tells us that the solution is what we get if we try to find a fixed point when iterating the use of this bound.

Proof. Note that it suffices to prove the inequality

$$E(t) \le E_0(1+\delta) \exp\left(\int_0^t r(t')\, dt' \right)$$

on $[0,T]$ for all δ. Also note that for some T_0, this inequality holds on $[0,T_0]$ by continuity.

Now assume that

$$E(t) \le E_0(1+\delta) \exp\left(\int_0^t r(t')\, dt' \right)$$

on $[0,T]$. If we plug this bound into the iteration, we get

$$\begin{aligned} E(t) &\le E_0 + E_0(1+\delta) \int_0^t r(t') \exp\left(\int_0^{t'} r(t'')\, dt'' \right) dt' \\ &= E_0 + E_0(1+\delta) \left(\exp\left(\int_0^t r(t')\, dt' \right) - 1 \right) \\ &= E_0(1+\delta) \exp\left(\int_0^t r(t')\, dt' \right) - \underbrace{\delta E_0}_{>0}. \end{aligned}$$

This means that this bound we assumed holds on $[0, T' + \varepsilon]$ for some ε. The result now holds by trying to do this with the supremum of all T' such that this inequality holds on $[0, T']$. We get that this supremum must be T. □